ADMISSIBLE FUNCTIONS WITH MULTIPLE DISCONTINUITIES

BY

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ABSTRACT

Let T be the unit circle, α irrational and $F : T \to R$ a step function. A necessary and sufficient condition for the skew of the α-rotation by f (considered as taking values mod 1) to be minimal is given. Also, the boundedness of $\Sigma_{i=1}^{n} f(x + i\alpha) - n \int_{T} f(z)dz$ as $n \to \alpha$ is resolved.

1. Introduction

The concept of an admissible function has been very useful in the analysis of the dynamical nature of skew products (cf. [4], [3], [1]). One of the restrictions imposed on these functions is that there be a point of discontinuity whose orbit does not intersect the set of discontinuities at any other point. The purpose of this paper is to replace this condition by a more careful analysis of the net effect of the discontinuities along orbits. We shall work on the circle group, where admissible functions have been of greatest use.

Consider $\boldsymbol{T} = [0, 1)$ endowed with the compact group structure obtained from identification with $\mathbf{R}/\mathbf{Z}$, and let $\alpha \in \boldsymbol{T}$ be irrational (α shall at times also be considered as an element of $\mathbf{R}$). Let $f : \boldsymbol{T} \to \mathbf{R}$ be a step function, i.e. locally constant with a finite number of jump discontinuities. We shall adopt the convention that all step functions are continuous from the right. Define $\delta : \delta_f : \boldsymbol{T} \to \mathbf{R}$ by

$$\delta(x) = f(x^{+}) - f(x^{-}) = \lim_{\varepsilon \downarrow 0} f(x + \varepsilon) - f(x - \varepsilon),$$

and let $D = D_f = \{x \in \boldsymbol{T} \mid \delta(x) \neq 0\}$. Since D is finite,

$$\Delta(x) = \Delta_f(x) = \sum_{i=-\infty}^{\infty} \delta_f(x + i\alpha)$$

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is well defined for every $x \in \boldsymbol{T}$. Finally, for $n > 0$ set $F_n(x) = \sum_{i=0}^{n-1} f(x + i\alpha)$.

The first and simpler of the two questions we deal with is the boundedness of $\{F_n(x) - n \int_T f(z)dz\}_{n=1}^{\infty}$ for some/every $x \in \boldsymbol{T}$. The case $f(x) = 1_{[0,\beta]}(x)$ was first treated by Kesten, and $f(x) = 1_{[0,\beta]}(x) - 1_{[\gamma,\gamma+\beta]}(x)$ by Furstenberg, Keynes and Shapiro [1]. In fact, we have

Theorem A. *$\{F_n(x) - n \int_T f(z)dz\}_{n=1}^{\infty}$ is bounded for some/every $x \in \boldsymbol{T}$ if and only if* $\Delta(\cdot) \equiv 0$.

The theorems of Kesten and Furstenberg–Keynes–Shapiro follow as special cases of Theorem A.

To prove Theorem A we pass to the standard symbolic model in which f is made continuous. It is defined as follows: Let $A = \{v_1, \cdots, v_k\}$ be the set of values taken by f. Let $\Omega = A^{\mathbf{Z}}$ be endowed with the product topology, and $\sigma : \Omega \to \Omega$ be the left shift, i.e. $\sigma(s)[i] = s[i+1]$, $s \in \Omega$, $i \in \mathbf{Z}$. Define $\varphi : \boldsymbol{T} \to \Omega$ by $\varphi(x)[i] = f(x + i\alpha)$, and let $\tilde{\boldsymbol{T}} \subset \Omega$ be the closure of $\varphi(T)$.

Then $\tilde{\boldsymbol{T}}$ is invariant under σ, and $(\tilde{T}, \sigma)$ is a minimal flow. The last follows most easily from the *uniform* density of the orbits $x, x + \alpha, \cdots$, in T. Lastly, define $\tilde{f}(s) = s[0]$ and $\tilde{F}_n(s) = \sum_{i=0}^{n-1} \tilde{f}(\sigma^i s)$.

The proof of Theorem A is a straightforward generalization of the methods used for admissible functions with simple discontinuities. The next result however departs more radically from this vein.

In what follows and throughout this paper if $Z \subset \mathcal{G} \subset \mathbf{R}$ is a group, then $\mathcal{G}' \subset T$ denotes the factor group $\mathcal{G}/\mathbf{Z}$ considered as a subgroup of $\boldsymbol{T}$. Similarly $\forall r \in \mathbf{R}$, $r' \in \boldsymbol{T}$ is the coset $r + \mathbf{Z}$. $\mathcal{G}\alpha$ will denote the group scaled by a factor of α, unless $\mathcal{G} = \mathbf{Z}$, whence $(\mathbf{Z}\alpha)'$ is usually meant. Also, given any set of generators in $\mathbf{R}$ or $\boldsymbol{T}$, by the generated group we shall always mean the closed group generated by the set.

So let $E \subset \mathbf{R}$ be the group generated by the values of f and 1. Of course $E = \mathbf{R}$ unless f takes on only rational values, in which case E is a rational lattice. Set $X = \tilde{\boldsymbol{T}} \times E'$ and define $T : X \to X$ by

$$T(s, y) = (\sigma s, y + \tilde{f}(s)').$$

We wish to determine the minimality of (X, T).

To this end let $G \subset \mathbf{R}$ be the group generated by $\{\Delta(z)\}_{z \in D} \cup \{1\}$. We define the *G-essential value* of f as the element of $\mathbf{R}/(G + G\alpha)$ given by

$$e(f) = \left(\int_T f(z)dz + \sum_{z + \mathbf{Z}\alpha \in T/\mathbf{Z}\alpha} z\Delta(z) \right) + G + G\alpha.$$

$e(f)$ is determined by picking $z_1, \cdots, z_l \in \mathbf{R}$ such that $z_i - z_j \notin \mathbf{Z}\alpha \mod 1$ for $i \neq j$, and $\forall z \in D$, $z - z_i \in \mathbf{Z}\alpha \mod 1$ for some i. Then $\int_T f(z)dz + \sum_{i=1}^{l} z_i \Delta(z_i)$ is independent of the choice of $z_1, \cdots, z_l$ up to an element of $G + G\alpha$, showing that $e(f)$ is well defined. To avoid overly cumbersome notation, we shall consider $e(f)$ as an element of $\mathbf{R}$, taking any representative. All calculations with $e(f)$ will of course be eventually free of this choice.

A necessary and sufficient condition for (X, T) to be minimal can now be given as

THEOREM B. *(X, T) is minimal if and only if for every proper closed subgroup $G \subseteq S \subsetneq E$, $e(f) \notin S + S\alpha$.*

The minimality condition can be less cleanly but more clearly stated as follows. If $E = \mathbf{R}$ then (X, T) is minimal if and only if either $G = \mathbf{R}$ or $e(f) \notin Q + Q\alpha$. In case E is a rational lattice, we shall see that one can always write $e(f) = a + b\alpha$, with $a, b \in E$. The condition is then that a, b and G generate E.

The paper is organized as follows. Theorem A is proved in section 2. Section 3 starts with Proposition C, which provides a decomposition of $f(\cdot)$ that makes clear the significance of $e(f)$. Theorem B is then proved, followed by some examples, the first of which being the demonstration that if f has only single discontinuities, then (X, T) is always minimal.

2. When is $\{F_n(x) - n\int_T f(z)dz\}_{n=1}^{\infty}$ bounded?

PROOF OF THEOREM A. We may assume without loss of generality that $\int_T f(z)dz = 0$. Then suppose first that $\Delta(z) = 0$ $\forall z \in T$. Let $K \geqq 0$ be minimal such that $(D + i) \cap D = \emptyset$ $\forall i \geqq K$. For $n \geqq 2K$ let $D_n \subset T$ be the set of points at which $F_n(\cdot)$ is discontinuous. Obviously $D_n \subset \bigcup_{i=0}^{n-1}(D - i\alpha)$. We claim in fact that $D_n \subset \bigcup_{i=0}^{k-1}(D - i\alpha) \cup \bigcup_{i=n-k}^{n-1}(D - i\alpha)$. For this it suffices to show that $D_n \cap \bigcup_{i=k}^{n-k-1}(D - i\alpha) = \emptyset$. But if $x \in \bigcup_{i=k}^{n-k-1}(D - i\alpha)$, then $\sum_{i=0}^{n-1}\delta(x + i\alpha) = \sum_{i=-\infty}^{\infty}\delta(x + i\alpha) = 0$. Thus $x \notin D_n$.

The above shows that $F_n(\cdot)$ has at most $2K|D|$ discontinuities, the jumps at which belong to a finite set. Since $\int_T F_n(z)dz = 0$, this implies that the F_n's are uniformly bounded as desired.

Now suppose that $\{F_n(z)\}_{n=1}^{\infty}$ is bounded for some/every $z \in T$. Then $\{\tilde{F}_n(s)\}_{n=1}^{\infty}$ is bounded for every $s \in \tilde{T}$. A theorem of Gottschalk and Hedlund [2] then guarantees the existence of a continuous function $g : \tilde{T} \to \mathbf{R}$ satisfying

$$g(\sigma s) - g(s) = \tilde{f}(s) \qquad \forall s \in \tilde{T}.$$

Assume to the contrary that $\exists z \in \boldsymbol{T}$ with $\Delta(z) \neq 0$. Let $u, v \in \tilde{\boldsymbol{T}}$ be given by $u = \lim_{x \to z^-} \varphi(x)$ and $v = \lim_{x \to z^+} \varphi(x)$. Then

$$g(u) - g(v) = \lim_{n \to \infty} (g(\sigma^n u) - \tilde{F}_n(u)) - (g(\sigma^n v) - \tilde{F}_n(v))$$

$$= \lim_{n \to \infty} (g(\sigma^n u) - g(\sigma^n v)) + \tilde{F}_n(v) - \tilde{F}_n(u)$$

$$= 0 + \sum_{i=0}^{\infty} \delta(z + i\alpha).$$

The last equality follows since g is uniformly continuous on $\tilde{\boldsymbol{T}}$ and the distance between $\sigma^n u$ and $\sigma^n v$ approaches zero. Similarly,

$$g(u) - g(v) = \lim_{n \to \infty} (g(\sigma^{-n} u) + \tilde{F}_n(\sigma^{-n} v)) - (g(\sigma^{-n} v) + \tilde{F}_n(\sigma^{-n} v))$$

$$= \lim_{n \to \infty} (g(\sigma^{-n} v) - g(\sigma^{-n} v)) + \tilde{F}_n(\sigma^{-n} v) - \tilde{F}_n(\sigma^{-n} v)$$

$$= 0 - \sum_{i=-\infty}^{-1} \delta(z + i\alpha).$$

Therefore $\Sigma_{i=-\infty}^{\infty} \delta(z + i\alpha) = 0$, contradiction! □

If $f(x) = 1_{[0,\beta]}(x)$, then $\delta(0) = +1$, $\delta(\beta) = -1$ and $\delta(z) = 0$ elsewhere. Thus $\Delta(z) \equiv 0$ if and only if $\beta \in \mathbf{Z}\alpha$, yielding Kesten's theorem. Likewise, if $f(x) = 1_{[0,\beta]}(x) - 1_{[\gamma,\gamma+\beta]}(x)$, then $\Delta(z) \equiv 0$ if and only if either $\beta \in \mathbf{Z}\alpha$ or $\gamma \in \mathbf{Z}\alpha$. Thus also the theorem of Furstenberg, Keynes and Shapiro is obtained as a special case of Theorem A. It was in fact in [1] that an admissible function with two discontinuities in an orbit was first considered.

It is interesting to note that in case $\{F_n(x)\}_{n=1}^{\infty}$ is bounded, i.e. $\Delta(\cdot) \equiv 0$ (and $\int_{\boldsymbol{T}} f(z)dz = 0$), we can in actuality explicitly construct the solution $l : \boldsymbol{T} \to \mathbf{R}$ to

$$f(\cdot) \equiv l(\cdot + \alpha) - l(\cdot)$$

that pushes forward to the unique, up to an additive constant, continuous function $\tilde{l} : \tilde{\boldsymbol{T}} \to \mathbf{R}$ satisfying $\tilde{l}(\sigma s) - \tilde{l}(s) \equiv \tilde{f}(s)$; i.e. the theorem of Gottschalk and Hedlund may be concretely realized.

For let $d(z) = \Sigma_{i=0}^{\infty} \delta(z + i\alpha)$. Since $\Delta(\cdot) \equiv 0$, $d(z)$ vanishes for all but finitely many $z \in \boldsymbol{T}$. Set $s = \Sigma_{z \in \boldsymbol{T}} d(z)$. Then let $l(\cdot)$ be right continuous with constant slope s and jump $-d(z)$ at any $z \in \boldsymbol{T}$. This definition is valid since the sum of jumps is cancelled by the change due to slope.

Now $l(\cdot + \alpha) - l(\cdot)$ is a step function, with mean value zero. It remains to show only that it has the same discontinuities as $f(\cdot)$. But at any $z \in \boldsymbol{T}$, $l(\cdot+\alpha)-l(\cdot)$ has jump $-d(z+\alpha)+d(z)=-\sum_{i=1}^{\infty}\delta(z+i\alpha)+\sum_{i=0}^{\infty}\delta(z)=\delta(z)$, proving $l(\cdot+\alpha)-l(\cdot)\equiv f(\cdot)$ as desired. Since l is continuous outside a finite subset of $\bigcup_{i\in Z}(D+i\alpha)$, $l\circ\varphi^{-1}$ can be continuously extended to get the solution $\tilde{l}$ guaranteed by the theorem by Gottschalk and Hedlund.

3. The minimality of (X, T)

The first step in proving Theorem B is to reduce to an admissible function with single discontinuities:

PROPOSITION C. *There exist continuous functions* $\tilde{g}, \tilde{l}: \tilde{\boldsymbol{T}} \to \mathbf{R}$ *such that*

(1) $\tilde{g}(s)-e(f)\in G+G\alpha\ \forall s\in\tilde{\boldsymbol{T}}$, *where the* $G\alpha$ *component is independent of* s; *and*

(2) $\tilde{f}(s)\equiv\tilde{f}(\sigma s)-\tilde{l}(s)+\tilde{g}(s)$.

PROOF. Let $z_1,\cdots,z_l\in\mathbf{R}$ be representatives of $D+(\mathbf{Z}+\mathbf{Z}\alpha)$ such that $0\leqq z_1<z_2<\cdots<z_l<1$. Define $g:\boldsymbol{T}\to\mathbf{R}$ by

$$g(x)=\sum_{i=1}^{l}\Delta(z_i)1_{[z_i,1]}(x)+\left(\int_{T}f(z)dz-\sum_{i=1}^{l}\Delta(z_i)(1-z_i)\right).$$

Then $\delta_g(z_i)=\Delta(z_i)$ and $\delta_g(z)=0$ for $z\notin\{z_1,\cdots,z_l\}$. This holds also at zero since $\delta_g(0)=\Delta(0)+\sum_{i=1}^{l}\Delta(z_i)=\Delta(0)+\sum_{z\in T}\delta(z)=\Delta(0)$. Thus $\Delta_{f-g}(\cdot)\equiv 0$. Since also

$$\int_T g(z)dz=\int_T f(z)dz,$$

we can write

$$f(x)-g(x)\equiv l(x+\alpha)-l(x),$$

where $l(\cdot)$ is as in Section 2.

Now extend $g\circ\varphi^{-1}$ and $l\circ\varphi^{-1}$ continuously to $\tilde{g}$ and $\tilde{l}$. Property 1 follows since

$$g(x)-\left(\int_T f(z)dz+\sum_{i=1}^{l}z_i\Delta(z_i)\right)=\sum_{i=1}^{l}\Delta(z_i)(1_{[z_i,1]}(x)-1)$$

for all $x\in\boldsymbol{T}$. Property 2 is obvious. □

The significance of $e(f)$ can now be explained, as g must clearly be normalized to have mean value equal to that of f.

We can now proceed with

PROOF OF THEOREM B. Fix $s \in \tilde{T}$ and let $C \subset X$ be the closure of $\{T^n(s,0)\}_{n\in Z}$. For every $t \in \tilde{T}$, let $H'_t = \{g \in E' \mid (t,g) \in C\}$. $H' = H'_s$ is directly seen to be a (closed) group, of which the various H'_t's are then cosets. Writing $H'_t = h(t) + H'$, we have that $h : \tilde{T} \to E'/H'$ is a continuous map satisfying

$$h(\sigma t) = h(t) + (\tilde{f}(t)' + H') \qquad \forall t \in \tilde{T}.$$

This argument can be found in [4].

We claim $G' \subset H'$. For suppose to the contrary that $\Delta(z)' \not\in H'$ for some $z \in T$. Let $u, v \in \tilde{T}$ be defined by $u = \lim_{x\to z^-} \varphi(x)$ and $v = \lim_{x \to z^+} \varphi(x)$. As in the proof of Theorem A, we then have

$$h(u) - h(v) = \sum_{i=0}^{\infty} \delta(z + i\alpha)' + H',$$

while also

$$h(u) - h(v) = -\sum_{i=-\infty}^{-1} \delta(z + i\alpha)' + H',$$

contradiction!

Now let $\tilde{g}$ and $\tilde{l}$ be as in the proposition. Set $\tilde{G}_n = \Sigma_{i=0}^{n-1} \tilde{g} \circ \sigma^i$. Pick any $e \in \text{Range}\, \tilde{g}$ (remember $e - e(f) \in G + G\alpha$) and let $V' \subset T$ be the set of limit points of ne' as $\sigma^n s$ approaches s. V' is a closed subgroup of T.

The heart of the proof is now exposed when we prove

$$V' + G' = H'.$$

For suppose first that $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ is such that $\sigma^{n_k} s \to s$ and $n_k e' \to v \in V'$. Then

$$\tilde{F}_{n_k}(s) = \tilde{l}(\sigma^{n_k} s) - \tilde{l}(s) + \tilde{G}_{n_k}(s).$$

Since

$$\tilde{G}_{n_k}(s) - n_k e' \in G',$$

we have

$$\tilde{F}_{n_k}(s)' \to v \bmod G'$$

($\tilde{l}$ is continuous so $\tilde{l}(\sigma^{n_k} s) - \tilde{l}(s) \to 0$). Therefore

$$v \in H' + G' = H'.$$

Similarly taking $\{m_k\}_{k=1}^{\infty} \subset \mathbf{N}$ such that $\sigma^{m_k} s \to s$ and $\tilde{F}_{m_k}(s)' \to g \in H'$, we have

$$g \in V' + G'.$$

Thus $V' + G' = H'$ as desired.

Since (X, T) is minimal if and only if $E' = H'$, it remains to check only when $V' + G'$ equals E'. Suppose first that $E = \mathbf{R}$. If $G = \mathbf{R}$ then of course $G' = E'$. Otherwise G is a rational lattice, and the condition is that $e \notin Q + Q\alpha$. This is equivalent to the condition desired.

Now suppose that E is a rational lattice. Write

$$e = a + b\alpha,$$

where $a, b \in E$. This decomposition (which is obviously unique since α is irrational) can be derived either from the containment $V' \subset E'$, or more directly as follows. Let

$$D = \{z_1^1, \cdots, z_{k_1}^1, z_1^2, \cdots, z_{k_2}^2, \cdots, z_1^l, \cdots, z_{k_l}^l\},$$

where $z_m^i - z_n^j \in \mathbf{Z}\alpha$ if and only if $i = j$. Referring to Example 1 below, we then have for some $c \in E$

$$\int_T f(z)dz + \sum_{i=1}^{l} z_1^i \Delta(z_1^i) = \left(-\sum_{i=1}^{l} \sum_{m=1}^{k_i} z_m^i \delta(z_m^i) + c\right) + \sum_{i=1}^{l} z_1^i \sum_{m=1}^{k_i} \delta(z_m^i)$$

$$= c + \sum_{i=1}^{l} \sum_{m=2}^{k_i} \delta(z_m^i)(z_1^i - z_m^i) \in E + E\alpha.$$

To finish, we now need only verify that V' is the group generated by a' and b'. As V' is just the set of accumulation points of $n(a + b\alpha)'$ as $n\alpha$ approaches zero mod 1 (either from the right alone or from both sides), this is a simple exercise in algebra left to the such inclined reader. □

EXAMPLE 1. Suppose now that f has only single discontinuities, that is $(D + i\alpha) \cap D = \varnothing\ \forall i \neq 0$. Let $D = \{z_1, \cdots, z_l\}$, where $0 \leqq z_1 < z_2 < \cdots < z_l < 1$. Setting $z_0 = 0$ and $z_{l+1} = 1$, let a_i be the value of f on $[z_i, z_{i+1})$ for $i = 0, \cdots, l$. Then

$$\int_T f(z)dz = \sum_{i=0}^{l} a_i(z_{i+1} - z_i) = \sum_{i=1}^{l} z_i(a_{i-1} - a_i) + a_l$$

$$= -\sum_{i=1}^{l} z_i \delta(z_i) + a_l.$$

Thus

$$e(f) = \int_T f(z)dz + \sum_{i=1}^{l} z_i \delta(z_i) = a_l.$$

Of course for such f Proposition C is trivial, from which $e(f) \in \text{Range} f$ is also clear. Therefore (X, T) is always minimal, as a_i and G necessarily generate E. Compare this with proposition 1.13.2 in [4], where minimality is proved if $\Delta(z) = \delta(z)$ generates E for some z.

EXAMPLE 2. Let $f(x) = \gamma 1_{[0,\beta]}(x)$, where $\beta = k\alpha \mod 1$. Then $\Delta(\cdot) \equiv 0$, $G = \{0\}$ and

$$e(f) = \int_T f(z)dz = \gamma\beta = \gamma k\alpha.$$

If γ is irrational then $E = \mathbf{R}$ and (X, T) is minimal if and only if $\gamma k\alpha \notin Q + Q\alpha$, i.e. $\gamma \notin Q + Q1/\alpha$. If $\gamma = p/q$ on the other hand, with $(p, q) = 1$, then $e(f) = pk\alpha/q$, so (X, T) is minimal if and only if $(k, q) = 1$. It is a matter of interest that these are also the conditions for the ergodicity of (X, T) (with respect to Haar measure).

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